Complement on geometric realisation

- Recall: $|\cdot| : sSet \longrightarrow Top$

  is the unique functor (up to natural iso)

  with
$$\begin{cases} * \ |\Delta^n| = \Delta^n_{top} \\ * \ |f| : \Delta^n_{top} \to \Delta^m_{top} \text{ is given by} \\ \quad \text{the unique linear map extending the} \\ \quad \text{one on vertices determined by } f. \\ * \ |\cdot| \text{ commutes with small colimits.} \end{cases}$$

- Let $X \in sSet$. Then we know

$$X = \operatorname*{colim}_{\int X} \Delta^n \quad \left(\int X \text{ category of elements} \right)$$

$$\Rightarrow |X| = \operatorname*{colim}_{\int X} \Delta^n_{top}$$

Like any colimit in a cocomplete category, it can be written as a coequalizer of maps between coproducts; coproducts in Top are disjoint unions and coequalizers are quotients by an eq. relation.

$$\Rightarrow \quad |X| = \left(\coprod X_n \times \Delta^n_{top} \right) \Big/ \begin{array}{c} (\delta^*(x), t) \\ \wr \\ (x, \delta_*(t)) \end{array}$$

for $\delta \in \mathrm{Mor}(\Delta)$.

Moreover as we saw, the skeletal filtration shows that $|X|$ is a CW-complex.

- The geometric realisation has an additional very useful exactness property:

(Non-) theorem: $|\cdot|$ ~~commutes with finite limits~~.

Alas this is not quite true... but almost!

Thm: a) $|\cdot|$ commutes with equalizers.

b) Let $X, Y \in sSet$. The canonical map

$$|X \times Y| \longrightarrow |X| \times |Y|$$

is a bijection of sets, and is an homeomorphism whenever $|X|$ or $|Y|$ is locally compact (for instance, when X or Y has finitely many non-degenerate simplices) $\square$

Cor: Let $f, g : X \to Y$ in sSet. If f is simplicially homotopic to g, i.e.
$\exists\ H : X \times \Delta^1 \to Y$ with $\begin{cases} H|_{X \times \Delta^{\{0\}}} = f \\ H|_{X \times \Delta^{\{1\}}} = g \end{cases}$
then $|f|$ is homotopic to $|g|$.

proof: $|X| \times [0,1] \overset{\text{Thm}}{\cong} |X \times \Delta^1| \xrightarrow{|H|} |Y|$ gives the required homotopy. $\square$

Here is why the thm is more complicated: Top has a basic pathology which is inconvenient in many places in algebraic topology:

⚠ Top is not cartesian closed.
(i.e $A \times -$ does not have a right adjoint)

But it turns out this is not a problem in practice: there exists "convenient full subcategories" $Top' \subseteq Top$ which:

- contain CW-complexes
- are complete and cocomplete.
- are cartesian closed.
- are closed under closed subspaces: ($A \in Top'$, $B \subset A$ closed $\Rightarrow B \in Top'$)
- are such that if $A, B \in Top'$ and A is locally compact, then $A \times_{Top} B \in Top'$

Hence $A \times_{Top} B = A \times_{Top'} B$.

These conditions are satisfied eg for

- Top' = compactly generated spaces.
- Top' = compactly generated weakly Hausdorff spaces.

See the survey paper "The category of CGWH spaces" of Neil Strickland for details.

Let $\mathrm{Top}' \subseteq \mathrm{Top}$ be any full subcategory satisfying the above. We still write

$$|\cdot| : \mathrm{sSet} \longrightarrow \mathrm{CW} \subset \mathrm{Top}'$$

for the corestriction of the geom. real.

thm: $|\cdot| : \mathrm{sSet} \longrightarrow \mathrm{Top}'$ preserves finite limits.

proof: It suffices to do equalizers and finite products. The case of finite products is the most interesting and useful, so I only give that part of the proof. (for equalizers, see [Gabriel-Zisman, III.3.3])

- First, we show that the canonical map $|\Delta^p \times \Delta^q| \longrightarrow |\Delta^p| \times |\Delta^q|$ is an homeomorphism for all p, q.

It is a continuous map between CW-complexes with $< \infty$ cells ($\Rightarrow$ compact Hausdorff) so it is enough to show it is a bijection.

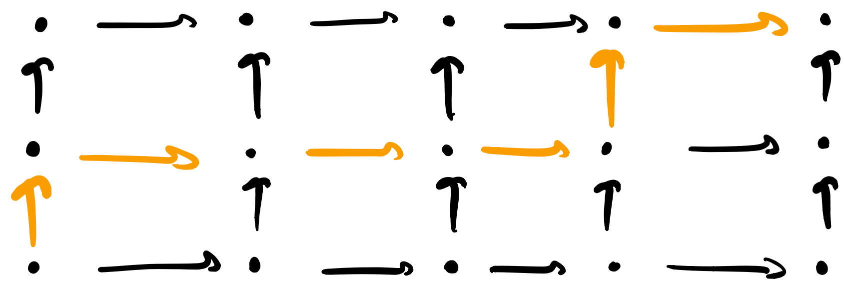
We sketch one combinatorial argument for this point (see [Gabriel-Zisman, II.5 and III.3] for details)

The poset $[p] \times [q]$ looks like:

$$
[2] \quad
\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u]
\end{tikzcd}
$$

$$[4]$$

The maximal chains (= totally ordered subsets) are all of length $p+q$ and look like:



There are $N = \binom{p+q}{q}$ such chains.

For a chain $C \cong [p+q]$, the projections onto $[p]$ and $[q]$ define maps

$$\begin{tikzcd}[row sep=small] & {[p]} \\ {[p+q]} \arrow[ur] \arrow[dr] & \\ & {[q]} \end{tikzcd},$$

Hence a $(p+q)$-element $x_C \in (\Delta^p \times \Delta^q)_{p+q}$

These are precisely the non-degenerate $(p+q)$-simplices of $\Delta^p \times \Delta^q$,

and this leads to a presentation

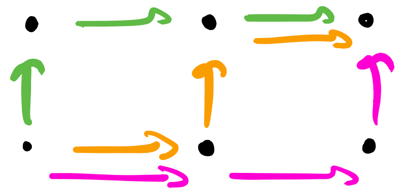
of $\Delta^p \times \Delta^q$ as an equalizer:

$$\coprod_{c \cap c'} \Delta^{|c \cap c'|} \rightrightarrows \coprod_{c} \Delta^{p+q} \xrightarrow{(x_c)} \Delta^p \times \Delta^q$$

We apply $|\cdot|$ and get

$$\coprod_{c \cap c'} \Delta_{top}^{|c \cap c'|} \rightrightarrows \coprod_{c} \Delta_{top}^{p+q} \longrightarrow |\Delta^p \times \Delta^q|$$

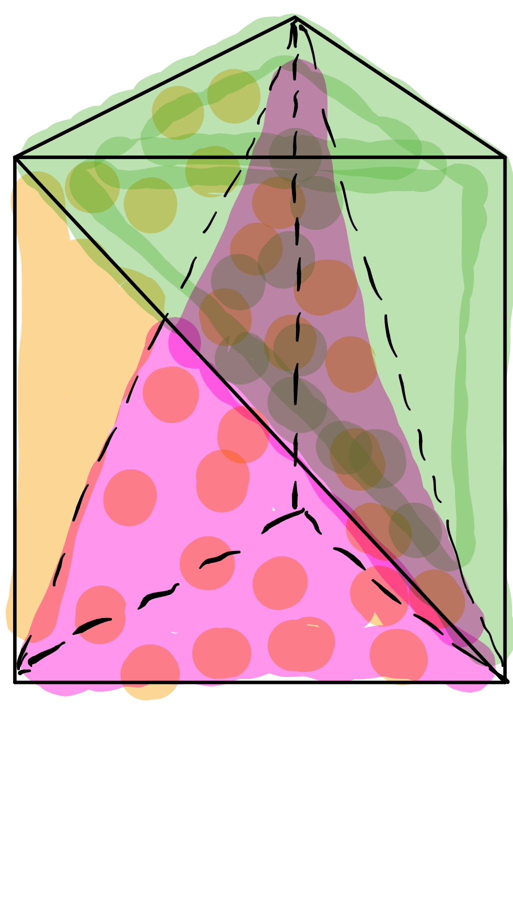
But the CW-complex $\Delta^p_{top} \times \Delta^q_{top}$ has a parallel presentation, induced by shuffle maps. At this point, I will just draw a picture and let you look at [Gabriel-Zisman] for details. Let $p=1$, $q=2$. Then $p+q=3$ and $N = \binom{3}{1} = 3$.

The maximal chains are 

The presentation of $\Delta^1 \times \Delta^2$ is

$$\Delta^1 \amalg \Delta^2 \amalg \Delta^2 \rightrightarrows \Delta^3 \amalg \Delta^3 \amalg \Delta^3 \to \Delta^1 \times \Delta^2$$

The geometric presentation of $\Delta^1_{top} \times \Delta^2_{top}$ is the decomposition of a prism into three pyramids :



Let's now do the general case.

The key point is that, in a cartesian closed category like Top' (but not Top!), finite products are left adjoints hence commute with colimits.

Let $X, Y \in sSet$. We write

$$X = \operatorname*{colim}_{\int X} \Delta^p, \quad Y = \operatorname*{colim}_{\int Y} \Delta^q$$

and we compute

$$|X \times Y| = \left| \operatorname*{colim}_{\int X} \Delta^p \times \operatorname*{colim}_{\int Y} \Delta^q \right|$$

$\times$ in sSet commutes with colim

$$\cong \left| \operatorname*{colim}_{\int X} \operatorname*{colim}_{\int Y} \Delta^p \times \Delta^q \right|$$

general fact about colimits

$$\cong \left| \operatorname*{colim}_{\int X \times \int Y} \Delta^p \times \Delta^q \right|$$

$|\cdot|$ commutes with colim

$$\cong \operatorname*{colim}_{\int X \times \int Y} |\Delta^p \times \Delta^q|$$

previous step

$$\cong \operatorname*{colim}_{\int X \times \int Y} (|\Delta^p| \times |\Delta^q|)$$

general fact about colim +

$\times$ commutes with colim in Top'

$$\cong \operatorname*{colim}_{\int X} |\Delta^p| \times \operatorname*{colim}_{\int Y} |\Delta^q|$$

$$\cong \left| \operatorname*{colim}_{\int X} \Delta^p \right| \times \left| \operatorname*{colim}_{\int Y} \Delta^q \right|$$

$|\cdot|$ commutes with colim.

$$\cong |X| \times |Y|$$

This (+ some verifications that this is indeed the canonical map) ends the proof. $\square$

2) ∞ - categories

def 5 An ∞-category (or quasicategory) is a simplicial set $C \in sSet$ satisfying the inner horn extension property: $\forall\, n \geq 2$, $\forall\, 0 < k < n$,

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

- A **functor** $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories is simply a morphism of simplicial sets. This defines a (1-) category Cat^1_∞ of ∞-categories: $\mathrm{Cat}^1_\infty \overset{\text{full}}{\hookrightarrow} \mathrm{sSet}$.

- A **natural transformation** $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd}$ is a morphism $\alpha : \mathcal{C} \times \Delta^1 \to \mathcal{D}$ with $\alpha|_{\mathcal{C} \times \{0\}} = F$ and $\alpha|_{\mathcal{C} \times \{1\}} = G$.

- A **natural isomorphism** $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd}$ is a natural transformation α such that there exists $\beta : G \Rightarrow F$ and maps $t, t' : \mathcal{C} \times \Delta^2 \to \mathcal{D}$ with

$$\begin{tikzcd} & G \arrow[dr, "\beta"] \arrow[d, phantom, "t"] & \\ F \arrow[ur, "\alpha"] \arrow[rr, "\mathrm{id}_F"'] & {} & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & F \arrow[dr, "\alpha"] \arrow[d, phantom, "t'"] & \\ G \arrow[ur, "\beta"] \arrow[rr, "\mathrm{id}_G"'] & {} & G \end{tikzcd}$$

- A functor $F : C \to D$ is a categorical equivalence (or an equivalence of ∞-categories) if there exists $G : D \to C$ and natural isomorphisms $\begin{cases} F \circ G \overset{\sim}{\Rightarrow} id_D \\ G \circ F \overset{\sim}{\Rightarrow} id_C \end{cases}$

Basic examples.

- By $\begin{cases} \text{the def. of Kan complexes} \\ \text{Prop 33 and Thm 34} \end{cases}$, we have fully faithful functors:

$$\begin{tikzcd} & \mathrm{Cat} \arrow[dr, hook, "N"] & & \\ \mathrm{Grp} \arrow[ur, hook] \arrow[dr, hook] & & \mathrm{Cat}_\infty^1 \arrow[r, hook] & s\mathrm{Set}. \\ & \mathrm{Kan} \arrow[ur, hook] & & \end{tikzcd}$$

- We will see later that there are many examples which are not of these forms.

With this terminology we can formulate the main goal of this course:

* Develop "category theory" for ∞-categories, by introducing categorical concepts like (co)limits, representable functors, adjoints, etc. in such a way that:

  - they are compatible with usual category theory via the nerve.
  - they are invariant under categorical equivalences.

This leads to, as a secondary goal:

* Study the homotopy theory of categorical equivalences and the "∞-category of ∞-categories".

Rmk: One could think at this point: "we are only going to work with ∞-categories and forget about general simplicial sets."

Not the case!

+ Λ^n_k is not an ∞-category.

+ We need to first construct, then check that something is an ∞-category.

+ Skeleta of ∞-cats are not ∞-cats, but still want to use skeletal filtration.

+ To define the ∞-category of ∞-categories, one uses a model structure on the whole sSet.

History

. This definition is due to Boardman-Vogt

(1973) in the context of homotopy theory (homotopy coherent algebraic structures, infinite loop spaces). They proved some basic results which we will review soon.

- The idea of taking quasicategories as a model for $(\infty,1)$-categories is due to Joyal (late 90's) and he developped most of the results from the first half of this course. Then Lurie came and pushed the theory even further!

Terminology For $X_\bullet \in sSet$ (and in particular for ∞-categories), we call

- objects of $X_\bullet$ the elements of X_0
- (1-)morphisms of $X_\bullet$ ———— X_1

For $f \in X_1$, we say that the source (resp. the target) of f is $d_1(f)$ (resp. $d_0(f)$) and we write $f: d_1(f) \longrightarrow d_0(f)$.

- For $x \in X_0$, we write $id_x = s_0(x)$ and call it the identity morphism of x.

A basic tool we want to have in any "category theory" is duality.

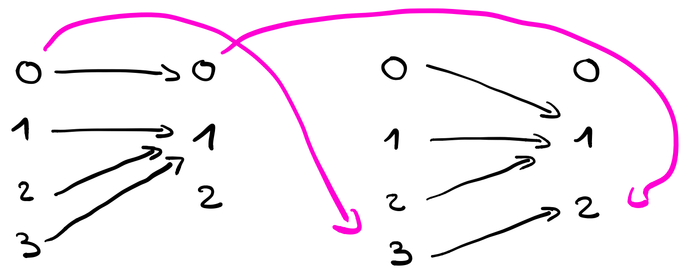
def 6 The order-reversing functor

$$\rho : \Delta \longrightarrow \Delta \text{ is defined as}$$

the identity on objects and, for $f:[m]\to[n]$,

$$\rho(f)(i) := n - f(m-i).$$

Ex:



f $\qquad\qquad$ $\rho(f)$

$\rho^* : sSet \longrightarrow sSet$ is the functor "precomposition by ρ". For $X_\cdot \in sSet$, the opposite simplicial set is $X_\cdot^{op} := \rho^*(X_\cdot)$. $\square$

Examples $\ast$ $(\Delta^n)^{op} \simeq \Delta^n$, $(I^n)^{op} \simeq I^n$, $(\partial\Delta^n)^{op} \simeq \partial\Delta^n$.

$\ast$ $(\Lambda^n_i)^{op} \simeq \Lambda^n_{n-i}$

$\ast$ $X_\cdot$ ∞-category (resp Kan) $\iff$ $X_\cdot^{op}$ ∞-category (resp Kan).

* $N(C^{op}) \cong N(C)^{op}$ for $C \in Cat$.

. Let us discuss some (co)limits of ∞-categories.

Prop 7: 1) Arbitrary products and coproducts of ∞-categories (in sSet) are ∞-categories.

2) Filtered colimits of ∞-categories are ∞-categories.

proof: 1)

Let $\{C_\alpha\}_{\alpha \in J}$ be a family of ∞-categories.

. Let $0 < k < n$. We have

$$\begin{array}{ccc} sSet(\Delta^n, \prod_\alpha C_\alpha) & \longrightarrow & sSet(\Lambda^n_k, \prod_\alpha C_\alpha) \\ \cong & & \cong \\ \prod_\alpha sSet(\Delta^n, C_\alpha) & \twoheadrightarrow & \prod_\alpha sSet(\Lambda^n_k, C_\alpha) \end{array}$$

product of surjections is a surjection

$\Rightarrow \prod_\alpha C_\alpha$ is an ∞-category.

- For the coproduct, we need to compute

$$\mathrm{sSet}\left(\Lambda^n_k, \coprod C_\alpha\right) \quad \text{and} \quad \mathrm{sSet}\left(\Delta^n, \coprod C_\alpha\right)$$

By Yoneda, $\mathrm{sSet}\left(\Delta^n, \coprod C_\alpha\right) = \left(\coprod_\alpha C_\alpha\right)([n])$

colimits are objectwise

$$= \coprod_\alpha C_\alpha([n])$$

$$= \coprod_\alpha \mathrm{sSet}(\Delta^n, C_\alpha)$$

So it suffices to show that the natural map

$$\coprod_\alpha \mathrm{sSet}\left(\Lambda^n_k, C_\alpha\right) \longrightarrow \mathrm{sSet}\left(\Lambda^n_k, \coprod_\alpha C_\alpha\right)$$

is a bijection. For this one can use

$$\Lambda^n_k = \coprod_{\substack{\Delta^{[n]\setminus\{i,j\}} \\ i \neq k}} \Delta^{[n]\setminus i}$$

and the Yoneda trick above: for any $X \in \mathrm{sSet}$,

$$\mathrm{sSet}\left(\Lambda^n_k, X\right) = \underset{\mathrm{sSet}(\Delta^{[n]\setminus\{i,j\}}, X)}{\times} \mathrm{sSet}\left(\Delta^{[n]\setminus i}, X\right).$$

The key point is that the various $\Delta^{[n]\setminus i}$ must be sent to the same C_α because they are connected via the $(n-2)$-faces $\Delta^{[n]\setminus\{i,j\}}$. Or in other words, one can show that

$$\pi_0\left(\coprod C_\alpha\right) \simeq \coprod_\alpha \pi_0(C_\alpha) \text{ while } \pi_0(\Lambda^n_k) \text{ has}$$

one element (Λ^n_k is connected).

2) Recall: a filtered category J is a (1-)category such that:

- $\forall\, i, j \in Ob(J)$, $\exists\, k \in Ob(J)$ and morphisms $i \to k$, $j \to k$
- $\forall\, i, j \in Ob(J)$, $f, g : i \rightrightarrows j$, then $\exists\, h : j \to k$, $h \circ f = h \circ g$.

<u>Ex</u>: · $(\mathbb{N}, \leq)$ is filtered. | $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow X_3$ colim $= \bigcup X_n$

· any totally ordered set is filtered.

<u>prop:</u> • Filtered colimits in Set commute with finite limits.

• A category J is filtered iff colimits of diagrams $J \to \mathrm{Set}$ commute with finite limits.

Let J be a filtered category and $C : J \longrightarrow \mathrm{sSet}$ be a diagram so that each C_α is an ∞-category. Once again we have $\left(\operatorname{colim}_J C\right)_n = \operatorname{colim}_J C([n])$ because colimits are computed objectwise.

We want to show that the canonical map

$$\operatorname*{Colim}_{\alpha \in J} \mathrm{sSet}(\Lambda^n_k, C_\alpha) \longrightarrow \mathrm{sSet}(\Lambda^n_k, \operatorname*{Colim}_J C)$$

is a bijection. We will show this holds for Λ^n_k replaced by any $Y \in \mathrm{sSet}$ with finitely many non-degenerate simplices.

Let $\mathcal{C} = \{ Y \in sSet \mid sSet(Y,-) \text{ commutes with filtered colimits} \}$

As remarked above, $\Delta^n \in \mathcal{C}$ for all $n \in \mathbb{N}$.

Let's show that $\mathcal{C}$ is closed under finite colimits. Let K be a finite category and $Y : K \to \mathcal{C}$ be a diagram.

We have

$$\begin{array}{ccc}
\operatorname{Colim}_{\alpha \in J} sSet\left(\operatorname{Colim}_K Y, C_\alpha\right) & \longrightarrow & sSet\left(\operatorname{Colim}_K Y, \operatorname{Colim}_J C\right) \\
\wr\!\!\mid \text{ colim prop} & & \wr\!\!\mid \text{ colim prop.} \\
\operatorname{Colim}_{\alpha \in J} \operatorname{Lim}_{\beta \in K} sSet(Y_\beta, C_\alpha) & & \operatorname{Lim}_{\beta \in K} sSet(Y_\beta, \operatorname{Colim}_J C) \\
 & \searrow\!\!\!\!\!\!\!\!\sim \qquad \qquad \sim\!\!\!\!\!\!\!\!\swarrow \; Y_\beta \in \mathcal{C} & \\
 & \operatorname{Lim}_{\beta \in K} \operatorname{Colim}_{\alpha \in J} sSet(Y_\beta, C_\alpha) &
\end{array}$$

Filtered colimits commutes with finite limits in Set.

$\Longrightarrow \operatorname{Colim}_K Y \in \mathcal{C}.$

- By the skeletal filtration,

$$\left\{ \begin{array}{c} \text{s. sets with } < \infty \\ \text{non-deg. simplices} \end{array} \right\} = \left\{ \begin{array}{c} \text{finite colimits of} \\ \text{standard simplices} \end{array} \right\} \subset \mathcal{C}$$

$\square$

Rmk: • The limits and colimits in the proposition have an important additional property (which we won't prove now): they are invariant under categorical equivalences! I.e, if we have categorical equivalences $C_\alpha \xrightarrow{\simeq} D_\alpha$ for all α, then $\prod_\alpha C_\alpha \simeq \prod_\alpha D_\alpha$.

• Other types of limits and colimits in the 1-category Cat^1_∞, when they exist, may not satisfy this property! The situation will be corrected with (co)limits in the ∞-category of ∞-categories Cat_∞.

Given what we know about nerves, the following seems reasonable:

Slogan: An ∞-category is like the nerve of a category, except that composition of chains of composable morphisms is only well-defined up to homotopy, and these homotopies are compatible in a precise way.

In particular, it should be possible to get a 1-category by identifying all those homotopies. This amounts to giving a simple description of the fundamental category $\tau X_\cdot$ when $X_\cdot$ is an ∞-category. This was achieved by Boardman-Vogt.

Def 8: Let $X_\bullet \in sSet$, $x, y \in X_0$. We say that $f, g : x \to y$ are **left homotopic** (resp. **right homotopic**), written $f \sim_\ell g$ (resp. $f \sim_r g$) if $\exists\, t \in X_2$ with:

$$\begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "id_x"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=3, "t"] & & y \end{tikzcd} \qquad \left(\text{resp. } \begin{tikzcd} & y \arrow[dr, "id_y"] & \\ x \arrow[ur, "g"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=3, "t"] & & y \end{tikzcd} \right)$$

$\square$

Left and right homotopy are not necc. equivalence relations; however we have

Lemma 9 Let C be an ∞-category. Then left and right homotopy coincide and is an equivalence relation.

Proof: Let $f, g, h : x \to y$ in $C(x, y)$.

We prove:

a) $f \sim_\ell f$.

b) $f \sim_\ell g$ and $g \sim_\ell h$ imply $f \sim_\ell h$.

c) $f \sim_\ell g$ implies $f \sim_r g$.

d) $f \sim_\ell g$ implies $g \sim_\ell f$.

a): $t := f_{001}$ works:

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=4, "f_{001}"] & & y \end{tikzcd}$$

.

b), c), d): They are proven in the same way:

- construct from the given 2-simplices a map $\Lambda^3_i \longrightarrow C$, with $i = \begin{cases} 1, & \text{for b) \& c)} \\ 2, & \text{for d)} \end{cases}$
- appeal to the inner horn extension property to get $\Delta^3 \to C$.
- restrict to the "new" 2-simplex.

It is easier in this case to display simplices so that the missing face is the back of the picture:

<u>For b):</u>

$$\begin{tikzcd}
x \arrow[dr, "id_x"] \arrow[dd, bend right, "id_x"'] \arrow[drr, bend left, "f"] \arrow[d, phantom, "x_{000}"] & & \\
& x \arrow[r, "g"] \arrow[dl, "id_x"] \arrow[ur, phantom, "f \underset{e}{\sim} g"] \arrow[dr, phantom, "g \underset{e}{\sim} h"] & y \\
x \arrow[urr, bend right, "h"'] & &
\end{tikzcd}$$

<u>For c):</u>

$$\begin{tikzcd}
x \arrow[dr, "id_x"] \arrow[dd, bend right, "g"'] \arrow[drr, bend left, "f"] \arrow[d, phantom, "g_{001}"] & & \\
& x \arrow[r, "g"] \arrow[dl, "g"] \arrow[ur, phantom, "f \underset{e}{\sim} g"] \arrow[dr, phantom, "g_{011}"] & y \\
y \arrow[urr, bend right, "id_y"'] & &
\end{tikzcd}$$

For d):

$$\begin{tikzcd}
x \arrow[dd, "id_x"'] \arrow[dr, "id_x"] \arrow[drr, bend left, "g"] \arrow[dr, phantom, shift left=6ex, "g_{001}"] & & \\
 \arrow[r, phantom, "x_{000}"] & x \arrow[r, "g"] \arrow[dr, phantom, "f \sim_e g"] & y \\
x \arrow[ur, "id_x"'] \arrow[urr, bend right, "f"'] & &
\end{tikzcd}$$

Finally:

+ a), b), d) $\Rightarrow$ $\sim_e$ equivalence relation.

+ c) + c)^{op} $\Rightarrow$ $\sim_e \; = \; \sim_r$ $\square$

In this case we write $f \simeq g$ for $f \underset{\sim_r}{\sim_e} g$.

and we write $[f]$ for the homotopy class of f in $C(x, y)$.

def 10 : Let $C \in Cat_\infty^1$, $f : x \to y$, $g : y \to z$ and $h : x \to z$. We say that h is a composition of f and g if there is $t \in C_2$ with

$$\begin{tikzcd}
 & y \arrow[dr, "g"] \arrow[d, phantom, "t"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"'] & \, & z
\end{tikzcd}$$ $\square$

Prop 11: In an ∞-category, compositions exist; their homotopy class is well-defined and depends only on the homotopy classes of the morphisms being composed.

proof: • The existence is simply the extension property for $\Lambda^2_1 \hookrightarrow \Delta^2$.

• Let $\begin{cases} f \simeq f' : x \to y \\ g \simeq g' : x' \to y' \end{cases}$ and let $\begin{cases} h & \text{be a composition of } f \text{ and } g \\ h' & \text{———————— } f' \text{ and } g'. \end{cases}$

We must prove that $h \simeq h'$. It is enough to treat separately the cases $f = f'$ and $g = g'$. By working in C^{op}, we reduce to $g = g'$. As before we construct a horn Λ^3_2, extend and restrict:

$$\begin{tikzcd}
x \arrow[dr, "f"] \arrow[dd, "id_x"'] \arrow[rrd, bend left, "h"] & & \\
& y \arrow[r, "g"] & z \\
x \arrow[ur, "f'"] \arrow[rru, bend right, "h'"'] & &
\end{tikzcd}$$

$\square$

Prop 12: The resulting composition on homotopy classes of morphisms is associative and unital.

proof: same method as Prop 11: left as exercise. $\square$

def 13 Let C be an ∞-category. Its homotopy category $hC \in Cat$ has:

$$\begin{cases} Ob(hC) = C_0 \\ hC(x,y) = C(x,y)/\simeq \end{cases}$$

By Prop 11-12, this is indeed a 1-category.

Prop 14: The construction of the homotopy category defines a functor $h: Cat^1_\infty \longrightarrow Cat$ which is a left adjoint to $N: Cat \longrightarrow Cat^1_\infty$.

($\Longleftrightarrow$ $\tau C \simeq hC$ naturally in C)

proof: . The functoriality follows from the fact that for $F: C \to D$, $g \simeq g'$ in $C \Rightarrow F(g) \simeq F(g')$ which is clear.

. Let C be an ∞-category. We construct a natural equivalence (in fact isomorphism) of categories $\tau C \xrightarrow[\varphi]{\sim} hC$.

Because of Prop 3, to define φ it is enough to define $\begin{cases} C_0 \longrightarrow \mathrm{Ob}\, hC \\ C_1 \longrightarrow \mathrm{Mor}\, hC \end{cases}$

satisfying some relations given by identities and C_2.

We put $C_0 = \mathrm{Ob}\, hC$

$$C_1 \longrightarrow \mathrm{Mor}\, hC, \quad \gamma \longmapsto [\gamma]$$

and the relations are satisfied by def 13.

- By construction, φ is $\begin{cases} \text{bijective on objects} \\ \text{surjective on morphisms.} \end{cases}$

It remains to show φ is faithful.

Because of the lifting property for $\Lambda^2_1 \hookrightarrow \Delta^2$, any morphism in τC can be written as $\overline{\gamma}$ for $\gamma \in C_1$. Now suppose that $\gamma, \gamma' : x \to y$ satisfy $[\gamma] = [\gamma']$. By definition, there is an homotopy

$$\begin{tikzcd} x \arrow[d, equal] \arrow[dr, "\gamma'"] & \\ x \arrow[r, "\gamma"'] & y \arrow[ul, phantom, "\sigma" near end] \end{tikzcd}$$

, but this also implies $\overline{\gamma'} = \mathrm{id}_x \circ \overline{\gamma'} = \overline{\gamma}$ in τC and we are done. $\square$